QUADRATIC A_1 BOUNDS FOR COMMUTATORS OF SINGULAR INTEGRALS WITH BMO FUNCTIONS

CARMEN ORTIZ-CARABALLO

ABSTRACT. For any Calderón–Zygmund operator T and any BMO function b we prove the following quadratic estimate

$$\|[b,T]\|_{L^p(w)} \leq c\|b\|_{BMO}(pp')^2[w]^2_{A_1}, \qquad 1<p<\infty,\, w\in A_1,$$

with constant $c=c(n,T)$ being the estimate optimal on p and the exponent of the weight constant. As an endpoint estimate we prove

$$w(\{x\in\mathbb{R}^n : |[b,T]f(x)|>\lambda\} \leq c\,\Phi([w]_{A_1})^2\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right)w(x)\,dx,$$

where $\Phi(t)=t(1+\log^+ t)$ and constant $c=c(n,T,\|b\|_{BMO})$.

1. Introduction and main results

The purpose of this paper is to study A_1 weight estimates for commutators of singular integrals with BMO functions. Before stating the main results we will describe briefly some of the motivations and earlier developments.

In 1971, C. Fefferman and E.M. Stein [FS] established the following extension of the classical weak-type $(1,1)$ property of the Hardy-Littlewood maximal operator M:

$$\|M\|_{L^{1,\infty}(w)} \leq c\,[w]_{A_1}, \qquad w\in A_1 \tag{1.1}$$

Recall that w is an A_1 weight if there is a finite constant c such that

$$Mw\leq c\,w \quad \text{a.e.}$$

and we denote by $[w]_{A_1}$ the smallest of these c. In fact they proved something better:

$$\|Mf\|_{L^{1,\infty}(w)} \leq c\int_{\mathbb{R}^n}|f|\,Mw\,dx, \qquad w\geq 0. \tag{1.2}$$

which was used to derive vector-valued extensions of the classical estimates for M.

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It was conjectured by B. Muckenhoupt and R. Wheeden in the seventies that the analogue of (1.1) holds for T, namely,

$$\|Tf\|_{L^{1,\infty}(w)} \leq c \int_{\mathbb{R}^n} |f| \, Mw dx, \qquad w \geq 0, \tag{1.3}$$

but has been disproved for T the Hilbert transform in a recent paper by María del Carmen Reguera and Christoph Thiele [RgT].

The best result in this sense can be found in [Pe3] where M is replaced by $M_{L(\log L)^\varepsilon}$ for any $\varepsilon > 0$ with new constant c_ε which blows up when $\varepsilon \to 0$. Furthermore, the following weaker variant of (1.3):

$$\|H\|_{L^{1,\infty}(w)} \leq c \, [w]_{A_1}, \qquad w \in A_1, \tag{1.4}$$

seems to be false (see [NRVV]).

Of course, the corresponding weak L^p type estimates have also a lot of interest. If we consider again the Hardy-Littlewood maximal function M, an application of the Marcinkiewicz Interpolating Theorem gives

$$\|M\|_{L^p(w)} \leq c_n p' \, [w]_{A_1}^{1/p} \qquad w \in A_1 \tag{1.5}$$

However, similar estimates for the Hilbert transform are more difficult to prove as shown by R. Fefferman and J. Pipher in [FPi]. In this paper the authors established the following result

$$\|H\|_{L^p(w)} \leq c_{n,p} \, [w]_{A_1} \qquad p \geq 2 \tag{1.6}$$

being the exponent of $[w]_{A_1}$ best possible. This estimate was improved in several directions by A. Lerner, S. Ombrosy and C. Pérez, in [LOPe1] and [LOPe3] (see also [LOPe2] for a dual problem). Indeed, this result was extended to any $1 < p < \infty$ and to any Calderón-Zygmund operator. Moreover an endpoint estimate close to the conjecture (1.4) was also obtained. To be more precise, they proved the following results.

THEOREM 1.1. [LOPe3] *Let T be a Calderón-Zygmund operator and let $1 < p < \infty$. Then*

$$\|T\|_{L^p(w)} \leq c \, pp' \, [w]_{A_1} \tag{1.7}$$

where $c = c_{n,T}$. Furthermore this result is optimal.

We estate now the corresponding result related to conjecture (1.4).

THEOREM 1.2. [LOPe3]

Let T be a Calderón-Zygmund operator. Then

$$\|T\|_{L^{1,\infty}(w)} \le c\,\Phi([w]_{A_1}), \tag{1.8}$$

where $c = c_{n,T}$ and where $\Phi(t) = t(1+\log^+ t)$.

These results were motivated by important works by S. Petermichl and A. Volberg [PetV] for the Ahlfors-Beurling transform and by S. Petermichl [Pet1, Pet2] for the Hilbert transform and the Riesz Transforms. In these papers it has been shown that if T is any of these operators, then

$$\|T\|_{L^p(w)} \le c_{p,n}\,[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}} \qquad 1<p<\infty, \tag{1.9}$$

where the exponent $\max\{1,\frac{1}{p-1}\}$ is best possible. Note that $A_1 \subset A_p$, and $[w]_{A_p} \le [w]_{A_1}$ (See Section 2 for the definition of $[w]_{A_p}$). Therefore (1.9) clearly gives the right exponent in (1.6) when $p \ge 2$. However, (1.9) cannot be used in order to get the sharp exponent in the range $1<p<2$, becoming the exponent worst when p gets close to 1. We note also that the proofs in [Pet1, Pet2, PetV] are based on the Bellman function techniques for $p=2$. The case $p \ne 2$ follows by the sharp version of the extrapolation theorem of Rubio de Francia as can be found in [DGrPPet] , and it is not clear whether they can be extended to the wider class of Calderón-Zygmund operators as is done in [LOPe1, LOPe3] in the case of A_1. We remit the reader to [Pe4] for a survey on this topic and to the papers [CrMPe1] and [CrMPe2] for a recent and different approach to these problems.

The sharp bound (1.9) for any Calderón-Zygmund operator T has been proved recently in [Hyt] by T. Hytönen. As before, it is enough to consider the case $p=2$ by the sharp extrapolation theorem. Hytönen's proof is based on approximating T by generalized dyadic Haar shift operators with good bounds combined with the key fact that to prove (1.9) it is enough to prove the weak type $(2,2)$ estimate with the same linear bound as proved in [PeTV]. A direct proof avoiding this weak $(2,2)$ reduction can be found in [HytPeTV]. A bit earlier, in [L2], the sharp $L^p(w)$ bound for T was obtained for values of p outside the interval $(3/2,3)$ and the proof is based on the corresponding estimates for the intrinsic square function. Even more recently, the bound (1.9) for any Calderón-Zygmund has been further improved by T. Hytönen and C. Pérez in [HytPe], where a portion of the A_p constant of w is replaced by the weaker A_∞ constant.

The main purpose of this paper is to prove similar estimates to Theorems 1.1 and 1.2 for commutators of singular integral operators T with BMO functions b. These

operators were introduced by Coifman, Rochberg and Weiss in [CRoW] and formally they are defined by

$$[b,T]f(x) = b(x)T(f)(x) - T(b\,f)(x) = \int_{\mathbb{R}^n} (b(x)-b(y))K(x,y)f(y)\,dy, \tag{1.10}$$

where K is a kernel satisfying the standard Calderón-Zygmund estimates (see Section 2 for the precise definition). Although the original interest in the study of such operators was related to generalizations of the classical factorization theorem for Hardy spaces many other applications have been found and in particular in partial differential equations.

The main result from [CRoW] states that $[b,T]$ is a bounded operator on $L^p(\mathbb{R}^n)$, $1<p<\infty$, when b is a BMO function. In fact, the BMO condition of b is also a necessary condition for the L^p-boundedness of the commutator when T is the Hilbert transform. These operators often behave as Calderón-Zygmund operators but there are some differences. For instance, an interesting fact is that, unlike what it is done with singular integral operators, the proof of the L^p-boundedness of the commutator does not rely on a weak type $(1,1)$ inequality. In fact, simple examples show that in general $[b,T]$ fails to be of weak type $(1,1)$ when $b \in BMO$. This was observed by Pérez in [Pe1] where it is also shown that there is an appropriate weak-$L(\log L)$ type estimate replacement (see below). Also it is shown by the same author in [Pe2] that M is not the right operator controlling $[b,T]$ but $M^2 = M \circ M$. These results amount to say that they behave differently from the Calderón-Zygmund operators.

In the present paper we pursue this point of view by showing that commutators have an extra "bad" behavior from the point of view of A_1 weights when trying to derive theorems such as Theorems 1.1 or 1.2. Related to the first theorem we have the following result.

THEOREM 1.3. *Let T be a Calderón–Zygmund operator and let b be in BMO. Also let $1<p,r<\infty$. Then there exists a constant $c=c_{n,T}$ such that for any weight w, we claim that the following inequality holds*

$$\|[b,T]f\|_{L^p(w)} \leq c\,\|b\|_{BMO}\,(pp')^2\,(r')^{1+\frac{1}{p'}}\,\|f\|_{L^p(M_r w)}. \tag{1.11}$$

In particular if $w \in A_1$, we have

$$\|[b,T]\|_{L^p(w)} \leq c\,\|b\|_{BMO}(pp')^2[w]_{A_1}^2. \tag{1.12}$$

Furthermore this result is sharp in p and in the exponent of $[w]_{A_1}$.

It should be mentioned that D. Chung proved in his dissertation (2010) that the commutator $[b, H]$, where H is the Hilbert transform and $b \in BMO$, obeys a quadratic bound in $L^2(w)$ with respect to the A_2 constant of the weight [Ch]. His proof is based on dyadic methods combined with Bellman functions techniques. On the other hand there is a new proof following an idea from [CRoW] by Chung-Pereyra-Pérez [ChPP]. This result is more general and states that if a linear operator T which obeys a linear bound in $L^2(w)$ with respect to the A_2 constant, then its corresponding commutator obeys a quadratic bound. In light of Hytönen's result this implies that all commutators of Calderón-Zygmund singular integral operators and BMO functions obey a quadratic bound in $L^2(w)$ which can then be extrapolated to $L^p(w)$. These results have been generalized by Cruz-Uribe and Moen [CrMo] to the two-weight setting, to fractional integrals, and to vector-valued extensions.

The second main result of this paper is the following endpoint version of Theorem 1.3.

THEOREM 1.4. *Let T and b as above. Then there exists a constant $c = c_{n,T,\|b\|_{BMO}}$ such that for any weight $w \in A_1$ and $f \in L^\infty_c(\mathbb{R}^n)$*

$$w(\{x \in \mathbb{R}^n : |[b,T]f(x)| > \lambda\} \le c\, \Phi([w]_{A_1})^2 \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) w(x)\, dx \tag{1.13}$$

where $\Phi(t) = t(1 + \log^+ t)$.

2. Preliminaries and notation

In this section we gather some well known definitions and properties which we will use along this paper.

Maximal operators. Given a locally integrable function f on $\mathbb{R}^n$, the Hardy–Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f(y), dy, \tag{2.1}$$

where the supremum is taken over all cubes Q containing the point x.

Also we will use the following operator:

$$M^\#_\delta f(x) = (M^\#(|f|^\delta)(x))^{1/\delta}$$

where $M^\#$ is the usual sharp maximal function of C. Fefferman-Stein:

$$M^\#(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy,$$

and as usual $f_Q = \frac{1}{|Q|}\int_Q f(y)\,dy$, and

$$M_\varepsilon f(x) = (M(|f|^\varepsilon)(x))^{1/\varepsilon}.$$

If the supremum is restricted to the dyadic cubes, we will use respectively the following notation M^d, $M_\delta^{\#,d}$ and M_δ^d.

Calderón–Zygmund operators. We will use standard well known definitions, see for instance [J, GrMF]. Let $K(x,y)$ be a locally integrable function defined of the diagonal $x=y$ in $\mathbb{R}^n\times\mathbb{R}^n$, which satisfies the size estimate

$$|K(x,y)| \le \frac{c}{|x-y|^n}, \tag{2.2}$$

and for some $\varepsilon>0$, the regularity condition

$$|K(x,y)-K(z,y)| + |K(y,x)-K(y,z)| \le c\frac{|x-z|^\varepsilon}{|x-y|^{n+\varepsilon}}, \tag{2.3}$$

whenever $2|x-z|<|x-y|$.

A linear operator $T : C_c^\infty(\mathbb{R}^n) \longrightarrow L^1_{loc}(\mathbb{R}^n)$ is a Calderón–Zygmund operator if it extends to a bounded operator on $L^2(\mathbb{R}^n)$, and there is a kernel K satisfying (2.2) and (2.3) such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy, \tag{2.4}$$

for any $f\in C_c^\infty(\mathbb{R}^n)$ and $x\notin supp(f)$.

We will use the following result from [APe] several times.

LEMMA 2.1. [APe] *Let T be a Calderón–Zygmund operator, and $0<\varepsilon<1$. Then there exists a constant c_ε such that*

$$M_\varepsilon^{\#}(Tf)(x) \le c_\varepsilon\, Mf(x). \tag{2.5}$$

Commutators. Let T be any operator and let b be any locally integrable function. The commutator operator $[b,T]$ is defined by

$$[b,T]f = b\,T(f) - T(bf).$$

As we already mentioned when T is any Calderón–Zygmund operator this operator is a bounded operator on $L^p(\mathbb{R}^n)$, $1<p<\infty$ ([CRoW]) but it is not of weak type $(1,1)$ when $b\in BMO$ and the following result holds [Pe1].

THEOREM 2.2. [Pe1] *Let b be a BMO function, $w \in A_1$ and T be a singular integral. Then there exists a positive constant $c = c_{\|b\|_{BMO},[w]_{A_1}}$ such that for all compact support function f and for all $\lambda > 0$*

$$w(\{x \in \mathbb{R}^n : |[b,T]f(x)| > \lambda\}) \leq c_{\|b\|_{BMO},[w]_{A_1}} \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) w(x)\, dx,$$

being $\Phi(t) = t(1 + \log^+ t)$.

Later on, G. Pradolini and C. Pérez in [PePr] improved this result as follows: given $\varepsilon > 0$

$$w(\{x \in \mathbb{R}^n : |[b,T]f(x)| > \lambda\}) \leq c\,\Phi\left(\|b\|_{BMO}\right) \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) M_{L(\log L)^{1+\varepsilon}} w(x) dx.$$

where $\Phi(t) = t(1+\log^+ t)$. The constant c is independent of the weight w, f and $\lambda > 0$. The point here is that there is no condition on the weight w.

One of the important steps to prove the last result is to give a version of the following classical result due to Coifman and C. Fefferman [CF]: let T be any Calderón–Zygmund operator and let $0 < p < \infty$, then there exists a constant $c = c_{p,[w]_{A_\infty}}$ such that for any $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \leq c_{p,[w]_{A_\infty}} \int_{\mathbb{R}^n} Mf(x)^p w(x) dx. \tag{2.6}$$

The corresponding version for commutators was found in [Pe2].

THEOREM 2.3. [Pe2] *Let $0 < p < \infty$ and let $w \in A_\infty$. Then there exists a positive constant $c = c_{p,[w]_{A_\infty},\|b\|_{BMO}}$ such that*

$$\int_{\mathbb{R}^n} |[b,T]f(y)|^p\, w(y)\, dy \leq c_{p,[w]_{A_\infty},\|b\|_{BMO}} \int_{\mathbb{R}^n} M^2 f(y)^p\, w(y)\, dy. \tag{2.7}$$

Besides, we will need the following pointwise inequality for commutators:

LEMMA 2.4. [Pe1] *Let b inBMO and let $0 < \delta < \varepsilon$. Then there exists a positive constant $c = c_{\delta,\varepsilon}$ such that,*

$$M_\delta^{\#}([b,T]f)(x) \leq c\|b\|_{BMO} \left(M_\varepsilon(Tf)(x) + M^2 f(x)\right), \tag{2.8}$$

for all smooth functions f.

Orlicz maximal functions. We need some few definitions and facts about Orlicz spaces. (For more information, see Bennett and Sharpley [BS] or Rao and Ren [RRe]). A function $B : [0,\infty) \to [0,\infty)$ is a doubling Young function if it is continuous, convex

and increasing, if $B(0)=0$ and $B(t)\to\infty$ as $t\to\infty$, and if it satisfies $B(2t)\le CB(t)$ for all $t>0$.

Recall that we defined the localized Luxembourg norm by equation $M_A = M_{A(L)}$, where $M_{A(L)}$ denotes a maximal type function defined by the expression

$$M_{A(L)}f(x)=\sup_{Q\ni x}\|f\|_{A,Q},$$

where A is any Young function and $\|f\|_{A,Q}$ denotes the A–average over Q defined by means of the Luxemburg norm

$$\|f\|_{A,Q}=\inf\left\{\lambda>0:\frac{1}{|Q|}\int_Q A\left(\frac{|f|}{\lambda}\right)\,dx\le 1\right\}. \tag{2.9}$$

An equivalent norm which is often useful in calculations is the following see Rao and Ren [RRe, p. 69]:

$$\|f\|_{A,Q}\le\inf_{\mu>0}\left\{\mu+\frac{\mu}{|Q|}\int_Q A\left(\frac{|f|}{\mu}\right)\,dx\right\}\le 2\|f\|_{A,Q}. \tag{2.10}$$

Given a Young function A, $\bar{A}$ will denote the complementary Young function associated to A; it has the property that for all $t>0$,

$$t\le A^{-1}(t)\bar{A}^{-1}(t)\le 2t.$$

The property that we will be using is the following generalized Hölder inequality

$$\frac{1}{|Q|}\int_Q |fg|\le 2\|f\|_{A,Q}\|g\|_{\bar{A},Q}. \tag{2.11}$$

B.M.O. functions and John-Nirenberg inequality. Denote the following pair of complementary Young functions

$$\Phi(t)=t\,(1+\log^+ t)\ \text{ and }\ \Psi(t)=e^t-1,$$

defining the classical Zygmund spaces $L(\log L)$, and $\exp L$ respectively. The corresponding averages will be denoted by

$$\|\cdot\|_{\Phi,Q}=\|\cdot\|_{L(\log L),Q}\ \text{ and }\ \|\cdot\|_{\Psi,Q}=\|\cdot\|_{\exp L,Q}.$$

Using this notation and the generalized Hölder inequality we also get

$$\frac{1}{|Q|}\int_Q |f(x)\,g(x)|\,dx\le c\,\|f\|_{\exp L,Q}\,\|g\|_{L(\log L),Q}. \tag{2.12}$$

This inequality allows to write the following formula that will be used in this article:

$$\frac{1}{|Q|}\int_Q |b(y)-b_Q|\,f(y)\,dy\le c\,\|b\|_{BMO}\|f\|_{L(\log L),Q}. \tag{2.13}$$

for any function $b \in BMO$ and any non negative function f. This inequality follows from (2.12) and the John-Nirenberg inequality [J] for BMO functions: there are dimensional positive constants $c_1 < 1$ and $c_2 > 2$ such that

$$\frac{1}{|Q|}\int_Q \exp\left(\frac{c_1|b(y)-b_Q|}{\|b\|_{BMO}}\right) dy \leq c_2$$

which easily implies that for appropriate constant $c > 0$

$$\|b - b_Q\|_{\exp L,Q} \leq c\,\|b\|_{BMO}.$$

Weights. We say that a weight w satisfies the A_p condition for $1 < p < \infty$ if there exists a constant c such that for any cube Q,

$$\left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} = c < \infty.$$

We will denote as $[w]_{A_p}$ the smallest of these c.

Also we recall that w is an A_1 weight if there is a finite constant c such that $Mw \leq c\,w$ a.e., and where $[w]_{A_1}$ denotes the smallest of these c. Also as usual, we denote $A_\infty = \bigcup_{p \geq 1} A_p$.

We will use several well-know properties about the A_p weights. First, it follows from Hölder's inequality that if $w_1, w_2 \in A_1$, then $w = w_1 w_2^{1-p} \in A_p$, and

$$[w]_{A_p} \leq [w_1]_{A_1}[w_2]_{A_1}^{p-1}. \tag{2.14}$$

Second, if $0 < \delta < 1$, then $(Mf)^\delta \in A_1$ (see [CRo]), and $f \in L^1_{loc}(\mathbb{R}^n)$

$$[(Mf)^\delta]_{A_1} \leq \frac{c_n}{1-\delta} \tag{2.15}$$

Third, is the Fefferman-Stein inequality [FS] saying that for any weight w,

$$\|Mf\|_{L^p(w)} \leq c_n\, p'\, \|f\|_{L^p(Mw)} \quad (1 < p < \infty), \tag{2.16}$$

where, as usual, p' denotes the dual exponent of p, $p' = \frac{p}{p-1}$.

Sharp reverse Hölder's inequality. An important step toward the proof of Theorem 1.3 is the following lemma which gives a precise version of the following classical reverse Hölder's inequality: if $w \in A_1$, there are constants $r > 1$ and $c \geq 1$ such that

$$M_r w(x) \leq c w(x). \tag{2.17}$$

LEMMA 2.5. [LOPe1] *Let $w \in A_1$, and let $r_w = 1 + \frac{1}{2^{n+1}[w]_{A_1}}$. Then,*

$$M_{r_w} w(x) \leq 2[w]_{A_1} w(x). \tag{2.18}$$

Building A_1 weights from duality. The following lemma gives a way to produce A_1 weights with special control on the constant. It is based on the so called Rubio de Francia iteration scheme or algorithm.

LEMMA 2.6. [LOPe3] *Let $1 < s < \infty$, and let v be a weight. Then, there exists a nonnegative sublinear operator R bounded in $L^s(v)$ satisfying the following properties:*

(i) $h \leq R(h)$;
(ii) $\|Rh\|_{L^s(v)} \leq 2\|h\|_{L^s(v)}$;
(iii) $Rh\, v^{1/s} \in A_1$ *with*

$$[Rh\, v^{1/s}]_{A_1} \leq cs'. \tag{2.19}$$

The sharp maximal function with optimal constants. We will need the following result linking the weighted L^p norm of a function and its sharp maximal function. This result is central in our approach. It can be found in [Pe4].

LEMMA 2.7. [Pe4] *Let $0 < p < \infty$, $0 < \delta < 1$, and let $w \in A_q$, with $1 \leq q < \infty$. Then there exists a constant $c = c(n, q, \delta)$ such that*

$$\|f\|_{L^p(w)} \leq cp[w]_{A_q} \left\| M_\delta^{\#,d}(f) \right\|_{L^p(w)}, \tag{2.20}$$

for any function f such that $|\{x : |f(x)| > t\}| < \infty$.

Buckley's Theorem. The following well known result gives the sharp bound for the Hardy-Littlewood maximal function on $L^p(w)$.

THEOREM 2.8. [Bu] *Let M be the Hardy–Littlewood maximal function, $p > 1$ and $w \in A_p$. Then there exists a constant $c = c_n p'$ such that*

$$\|M\|_{L^p(w)} \leq c_n p' \,[w]_{A_p}^{\frac{1}{p-1}}. \tag{2.21}$$

S.M. Buckley did not get the constant $c_n p'$ that appears in (2.21) but, recently a simple and elegant proof of this theorem was given by Lerner [L1] where it was obtained.

Observe that we could simply use Theorem 2.8, inequality (2.14) and then (2.15) to show that $\|M\|_{L^p((M_r w)^{1-p})}$ is bounded by a multiple of $[(M_r w)^{1-p}]_{A_p}^{\frac{1}{p-1}} \leq [M_r w]_{A_1}^{\frac{p-1}{p-1}} \leq c\, r'$. However this estimate is worse than the one obtained in the following lemma.

LEMMA 2.9. *Let $1 < p, r < \infty$, then*

$$\|Mf\|_{L^p((M_r w)^{1-p})} \leq c\, p'(r')^{\frac{1}{p}} \|f\|_{L^p(w^{1-p})}. \tag{2.22}$$

Proof of Lemma 2.9: This is a consequence of Lemma 3.4 from [LOPe1]. Indeed, this lemma yields

$$\|Mf\|_{L^p((M_r w)^{1-p})} \leq c\, p'(r')^{1-\frac{1}{rp'}} \|f\|_{L^p(w^{1-p})},$$

but $1 - \frac{1}{rp'} = \frac{1}{p'} + \frac{1}{p} - \frac{1}{rp'} = \frac{1}{p} + \frac{1}{r'p'}$ and hence $(r')^{1-\frac{1}{rp'}} = (r')^{\frac{1}{p}+\frac{1}{r'p'}} \leq c\,(r')^{\frac{1}{p}}$ since $t^{1/t} \leq e, \quad t \geq 1$.

□

A two weight inequality for Singular Integrals with optimal bounds. The last important lemma we will use can be seen as a dual version of Lemma 2.9 for Singular Integrals. It can be found essentially in [LOPe3] (see also [Pe4]).

LEMMA 2.10. [LOPe3] *Let T be a Calderón–Zygmund singular integral operator, and let $1 < p, r < \infty$. Then there is a constant $c = C_{n,T}$ such that*

$$\|Tf\|_{L^p(w)} \leq cp'\,(r')^{1/p'} \|f\|_{L^p(M_r w)} \quad w \geq 0.$$

As a corollary of this lemma we have the following estimate which was crucial to derive the main result from [LOPe3] and will be used in the proof of Theorem 1.4.

COROLLARY 2.11. [LOPe3] *Let T be a Calderón–Zygmund singular integral operator, and let $1 < p, r < \infty$. Then there is a constant $c = C_{n,T}$ such that*

$$\|Tf\|_{L^{1,\infty}(w)} \leq c(p')^p (r')^{p-1} \int_{\mathbb{R}^n} f(x)\, M_r w(x) dx. \tag{2.23}$$

Kolmogorov inequality. Finally, we will employ several times the well known Kolmogorov inequality. Let $0 < p < q < \infty$, then there is a constant $C = C_{p,q}$ such that for any measurable function f

$$\|f\|_{L^p(Q, \frac{dx}{|Q|})} \leq C\, \|f\|_{L^{q,\infty}(Q, \frac{dx}{|Q|})}. \tag{2.24}$$

See for instance [GrCF] p. 91, ex. 2.1.5.

3. The strong case

Before proving Theorem 1.3, we need the following two results. The first one is inspired by Lemma 2.7, and the second one is about the L^p boundeness of M^2 from $L^p(w^{1-p})$ to $L^p((M_r w)^{1-p})$:

LEMMA 3.1. *Let $0 < p < \infty$, $1 \leq q < \infty$, $0 < \varepsilon \leq 1$ and $w \in A_q$. Suppose that f is such that for each $t > 0$, $|\{x : f(x) > t\}| < \infty$. Then there is a constant $c = c_{n,q,\varepsilon}$ such that*

$$\big\|M_\varepsilon^d f\big\|_{L^p(w)} \leq c\, p[w]_{A_q} \big\|M_\varepsilon^{\#,d} f\big\|_{L^p(w)}. \tag{3.1}$$

Proof:

(In this proof, and for simplicity of notation we denote $M = M^d$ and similarly for the other operators).

In order to prove Lemma 3.1, we apply Lemma 2.7 to $M_\varepsilon f$ with $\delta = \varepsilon_0$, such that $0 < \varepsilon_0 < \varepsilon < 1$, and get

$$\|M_\varepsilon f\|_{L^p(w)} \leq cp[w]_{A_q} \left\| M^{\#}_{\varepsilon_0}(M_\varepsilon f)\right\|_{L^p(w)}. \tag{3.2}$$

Then, we will finish if we see that if $0 < \varepsilon_0 < \varepsilon < 1$

$$M^{\#}_{\varepsilon_0}(M_\varepsilon f)(x) \leq c\, M^{\#}_\varepsilon f(x), \tag{3.3}$$

where recall that if $f \geq 0$

$$M^{\#}_{\varepsilon_0} f(x) = M^{\#}(f^{\varepsilon_0})^{1/\varepsilon_0}(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f^{\varepsilon_0} - (f^{\varepsilon_0})_Q|\, dy, \right)^{1/\varepsilon_0}. \tag{3.4}$$

Now fix x and a dyadic cube Q with $x \in Q$. Hence

$$\frac{1}{|Q|} \int_Q \left|(M(f^\varepsilon))^{\varepsilon_0/\varepsilon}(y) - ((M(f^\varepsilon))^{\varepsilon_0/\varepsilon})_Q\right|\, dy \tag{3.5}$$

$$\leq 2\frac{1}{|Q|} \int_Q (M(f^\varepsilon))^{\varepsilon_0/\varepsilon}(y)\, dy - \inf_Q (M(f^\varepsilon))^{\varepsilon_0/\varepsilon} \tag{3.6}$$

adding and susbtracting $\inf_Q (M(f^\varepsilon))^{\varepsilon_0/\varepsilon}$.

Now we have,

$$f^\varepsilon(x) = g(x) + h(x), \tag{3.7}$$

where $g(x) = (f^\varepsilon(x) - f^\varepsilon_Q)\chi_Q(x)$ and $h(x) = f^\varepsilon_Q \chi_Q(x) + f^\varepsilon(x)\chi_{(Q)^c}(x)$.

Then, since $\varepsilon_0/\varepsilon < 1$,

$$\frac{1}{|Q|} \int_Q (M(f^\varepsilon))^{\varepsilon_0/\varepsilon}(y)\, dy \leq \frac{1}{|Q|} \int_Q (Mg)^{\varepsilon_0/\varepsilon}(y)\, dy + \frac{1}{|Q|} \int_Q (Mh)^{\varepsilon_0/\varepsilon}(y)\, dy \tag{3.8}$$

To finish, we study each separately. For the first one we use Kolmogorov's inequality (2.24) with $p = \varepsilon_0/\varepsilon < 1 = q$, and the fact that M is of weak-type $(1,1)$,

$$\frac{1}{|Q|} \int_Q (Mg)^{\varepsilon_0/\varepsilon}(y)\, dy \leq C_{\varepsilon,\varepsilon_0} \left(\frac{1}{|Q|} \int_Q |g(y)|\, dy \right)^{\varepsilon_0/\varepsilon} \tag{3.9}$$

$$= C_{\varepsilon,\varepsilon_0} \left(\frac{1}{|Q|} \int_Q |f^\varepsilon(y) - f^\varepsilon_Q|\, dy \right)^{\varepsilon_0/\varepsilon} \tag{3.10}$$

$$\leq C_{\varepsilon,\varepsilon_0} \,(M^{\#}_\varepsilon f(x))^{\varepsilon_0}. \tag{3.11}$$

This part is the bad term because the other term is less singular. Indeed, we claim the following

$$\frac{1}{|Q|}\int_Q (Mh)^{\varepsilon_0/\varepsilon}(y)\,dy \le (\inf_Q M(f^\varepsilon))^{\varepsilon_0/\varepsilon}.$$

Combining this inequality together with (3.11) and (3.5) we derive (3.2) concluding the proof of the lemma.

To prove the claim we recall $Mh(x) = \sup_{R\ni x}\frac{1}{|R|}\int_R h(y)\,dy$, where the supremum is taken over any dyadic cube R containing x, and we distinguish two types of cubes:

(1) let $R \subset Q$. In this case,

$$\frac{1}{|R|}\int_R h(y)\,dy = \frac{1}{|R|}\int_R f_Q^\varepsilon\,dy = f_Q^\varepsilon \le \inf_Q M(f^\varepsilon). \tag{3.12}$$

(2) $R \supset Q$ In this case

$$\begin{aligned}\frac{1}{|R|}\int_R h(y)\,dy &= \frac{1}{|R|}|R\cap Q| f_Q^\varepsilon + \frac{1}{|R|}\int_{R\cap Q^c} f^\varepsilon(y)\,dy\\ &= \frac{|Q|}{|R|} f_Q^\varepsilon + \frac{1}{|R|}\int_{R\cap Q^c} f^\varepsilon(y)\,dy\\ &= \frac{1}{|R|}\int_{R\cap Q} f^\varepsilon(x)\,dx + \frac{1}{|R|}\int_{R\cap Q^c} f^\varepsilon(x)\,dx\\ &= \frac{1}{|R|}\int_R f^\varepsilon(x)\,dx \le \inf_Q M(f^\varepsilon).\end{aligned}$$

So,

$$\frac{1}{|Q|}\int_Q (Mh)^{\varepsilon_0/\varepsilon}(y)\,dy \le \frac{1}{|Q|}\int_Q (\inf_Q M(f^\varepsilon))^{\varepsilon_0/\varepsilon}\,dy = (\inf_Q M(f^\varepsilon))^{\varepsilon_0/\varepsilon}. \tag{3.13}$$

□

PROPOSITION 3.2. *Let M^2 be the composition $M \circ M$, and let $1 < p, r < \infty$. Then, there is a constant c independent of r, p such that*

$$\|M^2 f\|_{L^p((M_r w)^{1-p})} \le c\,(p')^2 (r')^{1+1/p} \|f\|_{L^p(w^{1-p})}.$$

PROOF:

To prove the inequality we use Buckley's Theorem 2.8

$$\begin{aligned}\|M^2 f\|_{L^p((M_r w)^{1-p})} &= \|M(Mf)\|_{L^p((M_r w)^{1-p})}\\ &\le c_n p'\,[(M_r w)^{1-p}]_{A_p}^{1/(p-1)} \|Mf\|_{L^p((M_r w)^{1-p})}\\ &\le c_n p'\,[(M_r w)]_{A_1}^{(p-1)/(p-1)} \|Mf\|_{L^p((M_r w)^{1-p})}\end{aligned}$$

$$\leq c_n\, r'(p')^2 (r')^{1/p} \|f\|_{L^p(w^{1-p})}$$

where we have used property (2.14) and Lemma 2.9.

□

Proof of Theorem 1.3: We will prove (1.11), namely

$$\|[b,T]f\|_{L^p(w)} \leq c(pp')^2 \, \|b\|_{BMO} \, (r')^{1+\frac{1}{p'}} \, \|f\|_{L^p(M_r w)}. \tag{3.14}$$

A direct application of Lemma 2.5 with $r = r_w = 1 + \frac{1}{2^{n+1}[w]_{A_1}}$ would finish the proof of the Theorem 1.3.

By duality (3.14) is equivalent to proving

$$\left\| \frac{[b,T]^* f}{M_r w} \right\|_{L^{p'}(M_r w)} \leq c\,(pp')^2 \, \|b\|_{BMO} \, (r')^{1+\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}, \tag{3.15}$$

where $[b,T]^*$ is the adjoint operator of $[b,T]$ with respect to the L^2-pairing. Now,

$$\left\| \frac{[b,T]^* f}{M_r w} \right\|_{L^{p'}(M_r w)} = \sup_{\|h\|_{L^p(M_r w)} = 1} \left| \int_{\mathbb{R}^n} [b,T]^* f(x) h(x)\, dx \right|. \tag{3.16}$$

First by part (i) of Lemma 2.6 for $s = p'$ and $v = M_r w$ there exists an operator R such that

$$\begin{aligned} I &= \left| \int_{\mathbb{R}^n} [b,T]^* f(x) h(x)\, dx \right| \\ &\leq \int_{\mathbb{R}^n} |[b,T]^* f(x)| \, |h(x)| \, dx \\ &\leq \int_{\mathbb{R}^n} |[b,T]^* f(x)| \, Rh(x)\, dx. \end{aligned}$$

Besides, combining part (iii) of Lemma 2.6 for $s = p$ and $v = M_r w$ with (2.14) and (2.15) we obtain

$$[Rh]_{A_3} = [Rh(M_r w)^{1/p}(M_r w)^{-1/p}]_{A_3} = [Rh(M_r w)^{1/p}((M_r w)^{1/2p})^{-2}]_{A_3} \tag{3.17}$$

$$\leq [Rh(M_r w)^{1/p}]_{A_1} [(M_r w)^{1/2p}]_{A_1}^2 \tag{3.18}$$

$$\leq c_n\, p'. \tag{3.19}$$

Applying now Lemma 2.7 to $[b,T]^*$ with weight $w = Rh$, $q = 3$ and $p = 1$ we get

$$I \leq c_\delta \, [Rh]_{A_3} \int_{\mathbb{R}^n} M_\delta^{\#}([b,T]^* f)(x) \, Rh(x)\, dx.$$

Then we apply Lemma 2.4 with $0 < \delta < \varepsilon$ to $[b,T]^* = -[b,T^*]$ since is also a commutator with a Calderón-Zygmunnd operator. By Hölder's Inequality and using the property (ii) of the Lemma 2.6 and (3.19) we can continue with

$$\begin{aligned} I &\leq c_{n,\delta,\varepsilon}\|b\|_{BMO}p' \int_{\mathbb{R}^n} \left(M_\varepsilon(T^*f)(x) + M^2f(x)\right)\, Rh(x)\, dx \\ &= c_{n,\delta,\varepsilon}\|b\|_{BMO}p'(I_1 + I_2). \end{aligned}$$

Now

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^n} M^2f(x)\, Rh(x)\, dx \\ &\leq \left(\int_{\mathbb{R}^n} |M^2f(x)|^{p'}(M_rw(x))^{1-p'}\, dx\right)^{1/p'} \left(\int_{\mathbb{R}^n} M_rw(x)(Rh(x))^p\, dx\right)^{1/p} \\ &= 2\left\|\frac{M^2f}{M_rw}\right\|_{L^{p'}(M_rw)}. \end{aligned}$$

For the first term I_1, we apply Lemma 3.1 to $M_\varepsilon(T^*f)$ with weight $w = Rh$, $q = 3$ and $p = 1$, and then Lemma 2.1 (choosing now $0 < \varepsilon < 1$):

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n} |M_\varepsilon(T^*f)(x)|\,|Rh(x)|dx \leq c_{n,\varepsilon}\,[Rh]_{A_3}\int_{\mathbb{R}^n} |M_\varepsilon^{\#}(T^*f)(x)|\,|Rh(x)|dx \\ &\leq c_{n,\varepsilon}\,[Rh]_{A_3}\int_{\mathbb{R}^n} |Mf(x)|\,|Rh(x)|dx \leq c_{n,\varepsilon}\,p'\left(\int_{\mathbb{R}^n} Mf(x)^{p'}(M_rw(x))^{1-p'}\, dx\right)^{1/p'} \\ &= c_{n,\varepsilon}\,p'\left\|\frac{Mf}{M_rw}\right\|_{L^{p'}(M_rw)}, \end{aligned}$$

using property (ii) of Lemma 2.6 and Hölder's inequality, as we did when estimating I_2.

Therefore, combining estimates we have

$$\left\|\frac{[b,T]^*}{M_rw}\right\|_{L^{p'}(M_rw)} \leq c_{n,\delta,\varepsilon}\,\|b\|_{BMO}\,(p')^2\|M^2f\|_{L^{p'}((M_rw)^{1-p'})}. \tag{3.20}$$

Finally, to finish the proof of the theorem we apply Proposition 3.2 to get the estimate (3.15) which yields the claim (3.14).

□

4. The weak case

In this section we will prove Theorem 1.4. For $f \in C_0^\infty(\mathbb{R}^n)$ we consider the classical Calderón-Zygmund decomposition of f at level λ. Therefore we get a family $\{Q_j\} = Q_j(x_{Q_j}, r_j)$ of non-overlapping dyadic cubes satisfying

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|\,dx \le 2^n\lambda. \tag{4.1}$$

This implies that for $\Omega = \cup_j Q_j$ we have $|f(x)| \le \lambda$ a.e. on $\mathbb{R}^n \setminus \Omega$.

Now we split f in the standard "good" and "bad" parts $f = g + h$. Indeed, if as usual f_{Q_j} denotes the average of f on Q_j, we take

$$g(x) = \begin{cases} f(x), x \in \mathbb{R}^n \setminus \Omega \\ f_{Q_j}, \;\; x \in Q_j, \end{cases}$$

which also verifies $|g(x)| \le 2^n\lambda$ a.e. For the bad part we consider $h = \sum_j h_j$ where $h_j(x) = (f(x) - f_{Q_j})\chi_{Q_j}(x)$.

For each j we denote

$$\widetilde{Q_j} = 3Q_j, \quad \widetilde{\Omega} = \cup_j \widetilde{Q_j} \quad \text{and} \quad w_j(x) = w(x)\chi_{\mathbb{R}^n\setminus 3Q_j}$$

Then we can write

$$\begin{aligned} w(\{x \in \mathbb{R}^n : |[b,T]f(x)| > \lambda\}) \le\, & w(\{x \in \mathbb{R}^n \setminus \widetilde{\Omega} : |[b,T]g(x)| > \lambda/2\}) \\ & + w(\widetilde{\Omega}) \\ & + w(\{x \in \mathbb{R}^n \setminus \widetilde{\Omega} : |[b,T]h(x)| > \lambda/2\}) \\ & = I + II + III. \end{aligned}$$

and we study each term separately.

For the first one we use Chebyschev's inequality, calling $\widetilde{w}(x) = w(x)\chi_{\mathbb{R}^n\setminus\widetilde{\Omega}}(x)$, and by the L^p estimate (1.11) from Theorem 1.3 which holds for any weight we have

$$\begin{aligned} I &\le \frac{c}{\lambda^p}\int_{\mathbb{R}^n} |[b,T]g(x)|^p \widetilde{w}(x)\,dx \\ &\le \frac{c}{\lambda^p}\|b\|_{BMO}^p (pp')^{2p}(r')^{(1+\frac{1}{p'})p}\int_{\mathbb{R}^n} |g(x)|^p\, M_r\widetilde{w}(x)\,dx \\ &\le \frac{c}{\lambda}\|b\|_{BMO}^p (pp')^{2p}(r')^{2p-1}\int_{\mathbb{R}^n} |g(x)|\, M_r\widetilde{w}(x)\,dx \\ &= \frac{c}{\lambda}\|b\|_{BMO}^p (pp')^{2p}(r')^{2p-1}\left(\int_{\mathbb{R}^n\setminus\Omega} |f(x)|\, M_r\widetilde{w}(x)\,dx + \int_{\Omega} |g(x)|\, M_r\widetilde{w}(x)\,dx\right). \end{aligned}$$

It is clear that we only need to estimate the second term in the last expression and to do so we will be using the following fact: For any nonnegative function u with

$Mu(x) < \infty$ a.e., any cube Q, and any $R > 1$ we have

$$M(\chi_{\mathbb{R}^n \setminus RQ} u)(y) \approx M(\chi_{\mathbb{R}^n \setminus RQ} u)(z) \qquad y, z \in Q \tag{4.2}$$

with dimensional constants. This can be found in [GCRdF] p. 159.

Hence we can continue estimating the second term with

$$\begin{aligned}\int_\Omega |g(x)|\, M_r\widetilde{w}(x)\, dx &\le \sum_j \int_{Q_j} |f_{Q_j}|\, M_r(w_j)(x)\, dx \\ &= \sum_j \left(\int_{Q_j} |f(x)| dx \right) \frac{1}{|Q_j|} \int_{Q_j} M_r w_j(x)\, dx \\ &\le c \sum_j \left(\int_{Q_j} |f(x)|\, dx \right) \inf_{x \in Q_j} M_r w_j \\ &\le c \int_{\mathbb{R}^n} |f(x)|\, M_r w(x)\, dx.\end{aligned}$$

Up to now the parameter $1 < r < \infty$ was arbitrary but if we choose now $r = r_w$ where r_w is the sharp reverse Hölder exponent $r_w = 1 + \frac{1}{2^{n+1}[w]_{A_1}}$ from Lemma 2.5 we can continue with

$$\begin{aligned} I &\le \frac{c}{\lambda} \|b\|_{BMO}^p (pp')^{2p} [w]_{A_1}^{2p-1} \int_{\mathbb{R}^n} |f(x)|\, M_r w(x)\, dx \\ &\le \frac{c}{\lambda} \|b\|_{BMO}^p (pp')^{2p} [w]_{A_1}^{2p} \int_{\mathbb{R}^n} |f(x)| w(x)\, dx.\end{aligned}$$

The second term, II, is estimated in a very standard way

$$\begin{aligned} II = w(\widetilde{\Omega}) &\le c \sum_j \frac{w(\widetilde{Q_j})}{|\widetilde{Q_j}|} |Q_j| \\ &\le \frac{c}{\lambda} \sum_j \frac{w(\widetilde{Q_j})}{|\widetilde{Q_j}|} \int_{Q_j} |f(x)|\, dx \\ &\le \frac{c}{\lambda} \sum_j \int_{Q_j} |f(x)| Mw(x)\, dx \\ &\le \frac{c}{\lambda} \int_{\mathbb{R}^n} |f(x)| Mw(x)\, dx \qquad (4.3) \\ &\le \frac{c}{\lambda} [w]_{A_1} \int_{\mathbb{R}^n} |f(x)| w(x)\, dx, \qquad (4.4) \end{aligned}$$

because $w \in A_1$.

For part III first note that

$$[b,T]h(x)=\sum_j [b,T]h_j(x)=\sum_j (b(x)-b_{Q_j})Th_j(x)-\sum_j T((b-b_{Q_j})h_j)(x)$$

where as before $b_Q=\frac{1}{|Q|}\int_Q b$. Then

$$\begin{aligned}
III &\le w(\{x\in\mathbb{R}^n\setminus\widetilde{\Omega}: |\sum_j (b(x)-b_{Q_j})Th_j(x)|>\frac{\lambda}{4}\})\\
&\quad +w(\{x\in\mathbb{R}^n\setminus\widetilde{\Omega}: |\sum_j T((b-b_{Q_j})h_j)(x)|>\frac{\lambda}{4}\})\\
&= A+B.
\end{aligned}$$

Using the standard estimates of the kernel K we get

$$\begin{aligned}
A &\le \frac{c}{\lambda}\int_{\mathbb{R}^n\setminus\widetilde{\Omega}}\sum_j |b(x)-b_{Q_j}||Th_j(x)|\widetilde{w}(x)\,dx\\
&\le \frac{c}{\lambda}\sum_j\int_{\mathbb{R}^n\setminus 3Q_j}|b(x)-b_{Q_j}|w_j(x)\int_{Q_j}|h_j(y)||K(x,y)-K(x,x_{Q_j})|\,dy\,dx\\
&\le \frac{c}{\lambda}\sum_j\int_{Q_j}|h_j(y)|\int_{\mathbb{R}^n\setminus 3Q_j}|K(x,y)-K(x,x_{Q_j})||b(x)-b_{Q_j}|w_j(x)\,dx\,dy\\
&\le \frac{c}{\lambda}\sum_j\int_{Q_j}|h_j(y)|\sum_{k=1}^{\infty}\int_{2^kr_j\le|x-x_{Q_j}|<2^kr_j}\frac{|y-x_{Q_j}|^{\varepsilon}}{|x-x_{Q_j}|^{n+\varepsilon}}|b(x)-b_{Q_j}|w_j(x)\,dx\,dy\\
&\le \frac{c}{\lambda}\sum_j\int_{Q_j}|h_j(y)|\left(\sum_{k=1}^{\infty}\frac{2^{-k\varepsilon}}{(2^kr_j)^n}\int_{|x-x_{Q_j}|<2^{k+1}r_j}|b(x)-b_{Q_j}|w_j(x)\,dx\right)dy
\end{aligned}$$

To control the sum on k we use standard estimates together with the generalized Hölder inequality and John-Nirenberg's Theorem. Indeed if $y\in Q_j$ we have

$$\sum_{k=1}^{\infty}\frac{2^{-k\varepsilon}}{(2^{k+1}r_j)^n}\int\limits_{|x-x_{Q_j}|<2^{k+1}r_j}|b(x)-b_{Q_j}|w_j(x)\,dx$$

$$\begin{aligned}
&\le c\sum_{k=1}^{\infty}\frac{2^{-k\varepsilon}}{(2^{k+1}r_j)^n}\int_{2^{k+1}Q_j}|b(x)-b_{2^{k+1}Q_j}|w_j(x)\,dx\\
&\quad+\sum_{k=1}^{\infty}\frac{2^{-k\varepsilon}}{(2^{k+1}r_j)^n}\int_{2^{k+1}Q_j}|b_{2^{k+1}Q_j}-b_{Q_j}|w_j(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\leq c \sum_{k=1}^{\infty} 2^{-k\varepsilon} \|b - b_{2^{k+1}Q_j}\|_{\exp L, 2^{k+1}Q_j} \|w_j\|_{L\log L, 2^{k+1}Q_j} \\
&\quad + \sum_{k=1}^{\infty} 2^{-k\varepsilon}(k+1)\|b\|_{BMO}\, Mw_j(y) \\
&\leq c \sum_{k=1}^{\infty} 2^{-k\varepsilon}\|b\|_{BMO} M_{LlogL} w_j(y) + \sum_{k=1}^{\infty} 2^{-k\varepsilon}(k+1)\|b\|_{BMO}\,[w]_{A_1} w_j(y) \\
&\leq c \sum_{k=1}^{\infty} 2^{-k\varepsilon}(k+1)\|b\|_{BMO}[w]_{A_1} M_{LlogL} w_j(y).
\end{aligned}
$$

Then we can continue the estimate of A as follows

$$
\begin{aligned}
A &\leq \frac{c}{\lambda} \sum_j \int_{Q_j} |h_j(y)| \|b\|_{BMO}[w]_{A_1}\, M_{LlogL} w_j(y)dy \\
&\leq \frac{c}{\lambda}|\|b\|_{BMO}[w]_{A_1} \sum_j \left(\int_{Q_j} |f(y)|\, M_{LlogL} w_j(y)dy + \int_{Q_j} |f_{Q_j}|\, M_{LlogL} w_j(y)dy \right) \\
&\leq \frac{c}{\lambda}|\|b\|_{BMO}[w]_{A_1} \left(\int_{\mathbb{R}^n} |f(y)|\, M_{LlogL} w(y)dy + \sum_j \int_{Q_j} |f(x)|\, dx \frac{1}{|Q_j|} \int_{Q_j} M_{LlogL} w_j(y)dy \right) \\
&\leq \frac{c}{\lambda}|\|b\|_{BMO}[w]_{A_1} \left(\int_{\mathbb{R}^n} |f(y)|\, M_{LlogL} w(y)dy + \sum_j \int_{Q_j} |f(x)| \inf_{Q_j} M_{LlogL} w_j \right) \\
&\leq \frac{c}{\lambda}|\|b\|_{BMO}[w]_{A_1} \int_{\mathbb{R}^n} |f(y)|\, M_{LlogL} w(y)dy
\end{aligned}
$$

To estimate B we will use inequality (2.23) from Corollary 2.11 as follows

$$
\begin{aligned}
B &\leq \widetilde{w}(\{x \in \mathbb{R}^n : |T(\sum_j (b - b_{Q_j})h_j)(x)| > \frac{\lambda}{4}\}) \\
&\leq c\, \frac{(p')^p (r')^{p-1}}{\lambda} \int_{\mathbb{R}^n} \left| \sum_j (b(x) - b_{Q_j})h_j)(x) \right| M_r(\widetilde{w})(x)dx \\
&\leq c\, \frac{(p')^p (r')^{p-1}}{\lambda} \sum_j \int_{Q_j} |b(x) - b_{Q_j}||f(x) - f_{Q_j}|\, M_r w_j(x)dx \\
&\leq c\, \frac{(p')^p (r')^{p-1}}{\lambda} \sum_j \left(\int_{Q_j} |b(x) - b_{Q_j}||f(x)|\, M_r w_j(x)dx + \int_{Q_j} |b(x) - b_{Q_j}||f_{Q_j}|\, M_r w_j(x)dx \right) \\
&= c\,(p')^p (r')^{p-1}(B_1 + B_2) \leq c\,(p')^p [w]_{A_1}^{p-1}(B_1 + B_2)
\end{aligned}
$$

The estimate for B_2 we use (4.2)

$$\begin{aligned}
B_2 &= \frac{c}{\lambda}\sum_j \int_{Q_j} |b(x)-b_{Q_j}||f_{Q_j}|\, M_r w_j(x)dx \\
&\leq \frac{c}{\lambda}\sum_j \inf_{Q_j} M_r w_j \int_{Q_j} |b(x)-b_{Q_j}||f_{Q_j}|\, dx \\
&\leq \frac{c}{\lambda}\sum_j \frac{1}{|Q_j|}\int_{Q_j} |b(x)-b_{Q_j}| \int_{Q_j} |f(y)| M_r w_j(y)\, dy\, dx \\
&\leq c\|b\|_{BMO}\int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} M_r w(x)\, dx.
\end{aligned}$$

Observe that the constant c is dimensional.

For B_1 we have by the generalized Hölder inequality (2.11) and John-Nirenberg's theorem (2.13)

$$\begin{aligned}
B_1 &= \frac{c}{\lambda}\sum_j \int_{Q_j} |b(x)-b_{Q_j}||f(x)|\, M_r w_j(x)dx \\
&\leq \frac{c\|b\|_{BMO}}{\lambda}\sum_j \inf_{Q_j} M_r w_j |Q_j| \|f\|_{L\log L, Q_j}.
\end{aligned}$$

Now combining formula (2.10) together with (4.1) and recalling that $\Phi(t) = t(1+\log^+ t)$ we have

$$\begin{aligned}
\frac{1}{\lambda}|Q_j|\|f\|_{L\log L,Q_j} &\leq \frac{1}{\lambda}|Q_j| \inf_{\mu>0}\{\mu + \frac{\mu}{|Q_j|}\int_{Q_j}\Phi\left(\frac{|f(x)|}{\mu}\right)dx\} \\
&\leq |Q_j| + \int_{Q_j}\Phi\left(\frac{|f(x)|}{\lambda}\right)dx \\
&\leq \frac{1}{\lambda}\int_{Q_j}|f(x)|\,dx + \int_{Q_j}\Phi\left(\frac{|f(x)|}{\lambda}\right)dx \\
&\leq 2\int_{Q_j}\Phi\left(\frac{|f(x)|}{\lambda}\right)dx.
\end{aligned}$$

Then

$$\begin{aligned}
B_1 &\leq c\,\|b\|_{BMO}\sum_j \int_{Q_j}\Phi\left(\frac{|f(x)|}{\lambda}\right) w(x)dx \\
&\leq c\|b\|_{BMO}\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right) w(x)dx.
\end{aligned}$$

Combining this with estimates for I and II, we get that

$$w(x\in\mathbb{R}^n : |[b,T]f(x)|>\lambda) \le \frac{c\|b\|_{BMO}^p}{\lambda}(pp')^{2p}[w]_{A_1}^{2p}\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right)M_{LlogL}w(x)dx.$$

Setting here $p=1+\frac{1}{\log(1+\|w\|_{A_1})}$ gives the estimate we were looking for finishing the proof, because $\frac{1}{\log(1+\|w\|_{A_1})}<1$ and the fact that for every $A>1$, $A^{1/A}<e$.

□

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Departamento de Matemáticas, Escuela Politécnica, Universidad de Extremadura, Avda. Universidad, s/n, 10003 Cáceres, Spain

E-mail address: carortiz@unex.es